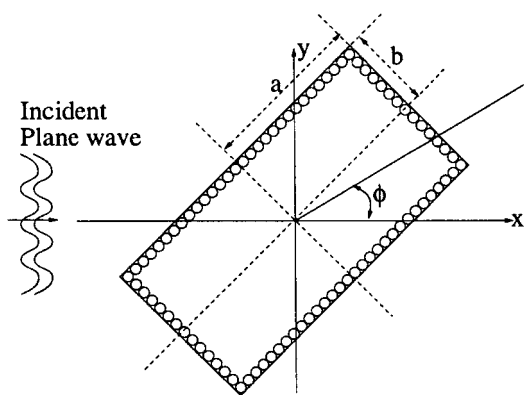
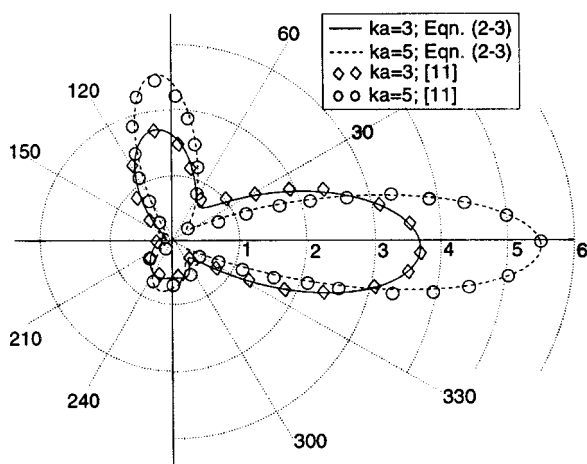




(a)



(b)

Figure 4 Comparison of normalized power density calculated using the recursive T -matrix algorithm with [11] (a) Tessellated geometry showing the scattering for 45° slanted metallic rectangle (b) Polar plot showing the normalized power density vs. ϕ for geometry of (a)

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COMPLEX COORDINATE STRETCHING AS A GENERALIZED ABSORBING BOUNDARY CONDITION

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ABSTRACT: By complex coordinate stretching and a change of variables, it is shown simply that PML is reflectionless for all frequencies and all angles. Also, Maxwell's equations for PML media reduce to ordinary Maxwell's equations with complex coordinate systems. Many closed-form solutions for Maxwell's equations map to corresponding closed-form solutions in complex coordinate systems. Numerical simulations with the closed-form solutions show that metallic boxes lined with PML media are highly absorptive. These closed-form solutions lend a better understanding to the absorptive properties of PML media. For instance, they explain why a PML medium is absorptive when a dielectric or metallic interface extends to the edge to a simulation region where PML media reside. More importantly, the complex coordinate stretching method can be generalized to non-Cartesian coordinate systems, providing absorbing boundary conditions in these coordinate systems. © 1997 John Wiley & Sons, Inc. *Microwave Opt Technol Lett* 15: 363–369, 1997.

Key words: absorbing boundary condition; perfectly matched layer; complex coordinate system

1. INTRODUCTION

The perfectly matched layer (PML) as a material absorbing boundary condition was first proposed by Berenger [1]. Later, Katz, Thiele, and Taflovie [2] extended it to three dimensions, while Chew and Weedon [3] proposed a coordinate stretching viewpoint. An anisotropic media interpretation has also been proposed by Sacks et al. [4]. Due to its efficacy as a material absorbing boundary condition, it has been fervently studied by many workers for numerical simulations of PDEs [5–13].

In this paper, we expand on the coordinate stretching viewpoint, and introduce a change of variables to transform Maxwell's equations in PML media into ordinary-looking Maxwell's equations in a complex coordinate system. As a result, many closed-form solutions that have already existed can be easily mapped into closed-form solutions in these complex coordinate systems. Moreover, the result can be easily generalized to non-Cartesian coordinate systems.

2. MODIFIED MAXWELL'S EQUATIONS

Maxwell's equations in a stretched coordinate system are given by [3]

$$\nabla_{\sigma} \times \mathbf{E} = i\omega\mu\mathbf{H} \quad (1)$$

$$\nabla_{\sigma} \times \mathbf{H} = -i\omega\epsilon\mathbf{E} \quad (2)$$

$$\nabla_{\sigma} \cdot \epsilon\mathbf{E} = 0 \quad (3)$$

$$\nabla_{\sigma} \cdot \mu\mathbf{H} = 0 \quad (4)$$

where

$$\nabla_{\sigma} = \hat{x} \frac{1}{s_x} \frac{\partial}{\partial x} + \hat{y} \frac{1}{s_y} \frac{\partial}{\partial y} + \hat{z} \frac{1}{s_z} \frac{\partial}{\partial z}. \quad (5)$$

In the above, s_i , $i = x, y, z$ are coordinate stretching variables. When s_i s, where $i = x, y, z$, are functions of i only, then (3) and (4) are derivable from (1) and (2).

A general plane-wave solution to (1)–(4) has the form

$$\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r}} \quad (6)$$

$$\mathbf{H} = \mathbf{H}_0 e^{i\mathbf{k} \cdot \mathbf{r}} \quad (7)$$

where $\mathbf{k} = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z$. In this case, (1) and (2) become

$$\mathbf{k}_{\sigma} \times \mathbf{E} = \omega\mu\mathbf{H} \quad (8)$$

$$\mathbf{k}_{\sigma} \times \mathbf{H} = -\omega\epsilon\mathbf{E} \quad (9)$$

where

$$\mathbf{k}_{\sigma} = \hat{x} \frac{k_x}{s_x} + \hat{y} \frac{k_y}{s_y} + \hat{z} \frac{k_z}{s_z}. \quad (10)$$

Combining the above, we have

$$-\omega^2\mu\epsilon\mathbf{H} = \mathbf{k}_{\sigma} \times \mathbf{k}_{\sigma} \times \mathbf{H} = \mathbf{k}_{\sigma}(\mathbf{k}_{\sigma} \cdot \mathbf{H}) - (\mathbf{k}_{\sigma} \cdot \mathbf{k}_{\sigma})\mathbf{H}. \quad (11)$$

But from (8), $\mathbf{k}_{\sigma} \cdot \mathbf{H} = 0$. Therefore,

$$\mathbf{k}_{\sigma} \cdot \mathbf{k}_{\sigma} = \omega^2\mu\epsilon \quad (12)$$

or

$$\omega^2\mu\epsilon = \chi^2 = \frac{1}{s_x^2}k_x^2 + \frac{1}{s_y^2}k_y^2 + \frac{1}{s_z^2}k_z^2. \quad (13)$$

Equation (13) is an equation of an ellipsoid in 3-D, and is satisfied by

$$k_x = \chi s_x \sin \alpha \cos \beta \quad (14)$$

$$k_y = \chi s_y \sin \alpha \sin \beta \quad (15)$$

$$k_z = \chi s_z \cos \alpha. \quad (16)$$

Equations (8) and (9) imply that $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{k}_{\sigma}$ are mutually orthogonal, and that the ratio of $|\mathbf{E}|$ to $|\mathbf{H}|$ is always

$$\eta = \frac{|\mathbf{E}|}{|\mathbf{H}|} = \frac{|\mathbf{k}_{\sigma}|}{\omega\mu} = \sqrt{\frac{\mu}{\epsilon}} \quad (17)$$

irrespective of the values of s_i , $i = x, y, z$. In (17), $|\mathbf{A}| = (\mathbf{A} \cdot \mathbf{A})^{1/2}$ where $\mathbf{A}$ can be a complex vector. Therefore, an interesting feature of this medium is that the wave impedance is independent of the stretching variables s_x , s_y , s_z . Moreover, s_x , s_y , and s_z can be complex, making k_x , k_y , and k_z complex. The k_i s can be independently controlled by different choices of s_x , s_y , and s_z to correspond to decaying waves in the x , y , and z directions.

3. REFLECTIONLESS ONE-DIMENSIONAL SOLUTION

Even though the one-dimensional PML problem has been proved to be reflectionless before [1, 3], a more succinct derivation is given here. The derivation of the previous section assumes that s_i , $i = x, y, z$ are constants of space. When s_i s are functions of i , $i = x, y, z$, we can derive a vector wave equation for inhomogeneous s_i from (1) and (2). It is given by

$$\nabla_{\sigma} \times \nabla_{\sigma} \times \mathbf{H} - \omega^2\mu\epsilon\mathbf{H} = 0. \quad (18)$$

We assume that μ and ϵ are constants. In this case, the above reduces to

$$\nabla_{\sigma} \cdot \nabla_{\sigma} \mathbf{H} + \omega^2\mu\epsilon\mathbf{H} = 0 \quad (19)$$

after invoking the condition that $\nabla_{\sigma} \cdot \mathbf{H} = 0$. Then, H_x , H_y , and H_z are independently solutions to

$$(\nabla_{\sigma} \cdot \nabla_{\sigma} + \omega^2\mu\epsilon)\phi = 0. \quad (20)$$

If we assume that $s_z(z)$ is a function of z only, and that s_x and s_y are constants equal to 1, the above reduces to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{s_z} \frac{\partial}{\partial z} \frac{1}{s_z} \frac{\partial}{\partial z} + \omega^2\mu\epsilon \right) \phi = 0. \quad (21)$$

Assuming that $\phi = \tilde{\phi}(z)e^{ik_x x + ik_y y}$, the above becomes

$$\left(\frac{1}{s_z} \frac{d}{dz} \frac{1}{s_z} \frac{d}{dz} + k_z^2 \right) \tilde{\phi}(z) = 0 \quad (22)$$

where $k_z^2 = \omega^2\mu\epsilon - k_x^2 - k_y^2$. Letting

$$\tilde{\phi}(z) = e^{\pm if(z)} \quad (23)$$

such that $f'(z) = k_z s_z(z)$ where k_z is a constant independent of z , then we have

$$\frac{1}{s_z} \frac{d}{dz} \tilde{\phi}(z) = \pm ik_z \tilde{\phi}(z) \quad (24a)$$

$$\frac{1}{s_z} \frac{d}{dz} \frac{1}{s_z} \frac{d}{dz} \tilde{\phi}(z) = -k_z^2 \tilde{\phi}(z) \quad (24b)$$

and hence (22) is satisfied. Therefore, the general solution to (22) is

$$\tilde{\phi}(z) = A e^{ik_z \int^z s(z') dz'} + B e^{-ik_z \int^z s(z') dz'}. \quad (25)$$

Notice that either one of the above two solutions can satisfy (22) independently. In other words, they are uncoupled, and each of them travels straight through the stretched medium without any reflection for all frequencies and all angles of incidence irrespective of the polarization!

4. GENERAL MULTIDIMENSIONAL SOLUTION

Motivated by the success of the above equation, we can derive a general solution when $s_x(x)$, $s_y(y)$, and $s_z(z)$ are, respectively, functions of x , y , z only. To this end, we propose a solution of the form

$$\phi(\mathbf{r}) = e^{i\mathbf{f}(\mathbf{r})} \quad (26)$$

where $\mathbf{f}(\mathbf{r}) = \hat{x}f_x(x) + \hat{y}f_y(y) + \hat{z}f_z(z)$. Then

$$\begin{aligned} \nabla_\sigma \phi(\mathbf{r}) &= i \left[\hat{x} \frac{1}{s_x} f'_x(x) + \hat{y} \frac{1}{s_y} f'_y(y) + \hat{z} \frac{1}{s_z} f'_z(z) \right] \phi(\mathbf{r}) \\ &= i \mathbf{k}_\sigma \phi(\mathbf{r}) \end{aligned} \quad (27)$$

where

$$\mathbf{k}_\sigma = \hat{x} \frac{1}{s_x} f'_x(x) + \hat{y} \frac{1}{s_y} f'_y(y) + \hat{z} \frac{1}{s_z} f'_z(z). \quad (28)$$

If we choose

$$f'_x(x) = \chi_x s_x, f'_y(y) = \chi_y s_y, f'_z(z) = \chi_z s_z \quad (29)$$

where χ_x, χ_y, χ_z are constants independent of space, then

$$\mathbf{k}_\sigma = \hat{x} \chi_x + \hat{y} \chi_y + \hat{z} \chi_z \quad (30)$$

which is a constant vector in space. Consequently,

$$\nabla_\sigma \cdot \nabla_\sigma \phi = -\mathbf{k}_\sigma \cdot \mathbf{k}_\sigma \phi = -\omega^2 \mu \epsilon \phi \quad (31)$$

where the last equality follows from (20). The above implies that

$$\mathbf{k}_\sigma \cdot \mathbf{k}_\sigma = \chi_x^2 + \chi_y^2 + \chi_z^2 = \omega^2 \mu \epsilon. \quad (32)$$

Therefore, the general solution (26) can be written as

$$\phi(\mathbf{r}) = e^{i \int^{\mathbf{r}} \mathbf{k}(\mathbf{r}') \cdot d\mathbf{r}'} \quad (33)$$

where we define

$$\begin{aligned} \int^{\mathbf{r}} \mathbf{k}(\mathbf{r}') \cdot d\mathbf{r}' &= \chi_x \int^x s_x(x') dx' \\ &+ \chi_y \int^y s_y(y') dy' + \chi_z \int^z s_z(z') dz'. \end{aligned} \quad (33a)$$

It is clear that (33) satisfies the Helmholtz wave equation in a general, inhomogeneously stretched coordinate system.

Since χ_x, χ_y , and χ_z can be positive or negative as long as the dispersion relation (32) is satisfied, Eq. (33) corresponds to waves propagating in all different directions. Moreover, a wave propagating in one given direction can exist independently of waves propagating in other directions. In other words, the waves are not coupled by inhomogeneous coordinate stretching, and hence, coordinate stretching does not induce reflections, irrespective of $s_x(x)$, $s_y(y)$, and $s_z(z)$, as long as they are, respectively, functions of x , y , z only.

5. MAXWELL'S EQUATIONS IN COMPLEX COORDINATES

Motivated by the above, we propose the following change of variables. Letting

$$\tilde{x} = \int_0^x s_x(x') dx' \quad (34a)$$

$$\tilde{y} = \int_0^y s_y(y') dy' \quad (34b)$$

$$\tilde{z} = \int_0^z s_z(z') dz', \quad (34c)$$

then the ∇_σ operator becomes

$$\tilde{\nabla} = \hat{x} \frac{\partial}{\partial \tilde{x}} + \hat{y} \frac{\partial}{\partial \tilde{y}} + \hat{z} \frac{\partial}{\partial \tilde{z}}. \quad (35)$$

The above follows because

$$\frac{\partial}{\partial \tilde{x}} = \frac{1}{s_x} \frac{\partial}{\partial x}, \frac{\partial}{\partial \tilde{y}} = \frac{1}{s_y} \frac{\partial}{\partial y}, \frac{\partial}{\partial \tilde{z}} = \frac{1}{s_z} \frac{\partial}{\partial z} \quad (36)$$

under the change of variables suggested by (34). In this case, Eqs. (1)–(4) just become ordinary-looking Maxwell's equations:

$$\tilde{\nabla} \times \mathbf{E} = i \omega \mu \mathbf{H} \quad (37)$$

$$\tilde{\nabla} \times \mathbf{H} = -i \omega \epsilon \mathbf{E} \quad (38)$$

$$\tilde{\nabla} \cdot \epsilon \mathbf{E} = 0 \quad (39)$$

$$\tilde{\nabla} \cdot \mu \mathbf{H} = 0 \quad (40)$$

except that $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ can be complex valued as well. As a result, many closed-form solutions can be obtained for PML media using the mapping provided by (34a)–(34c).

The interesting case occurs when $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ are complex. In this case, the waves traveling through the complex coordinates are attenuated, but not reflected. More importantly, the modified Maxwell's equations can be written in any coordinate system with complex variables as the coordinate variables. We will illustrate the solutions in the complex Cartesian, cylindrical, and spherical coordinates.

6. CARTESIAN COORDINATES

We will solve for the closed-form solution of a line source inside a metallic box of width a , and truncated in the vertical direction at $y = b$. The box is filled with two half spaces with different wavenumbers k_1 and k_2 , and the source is located at (x', y') , while the dielectric interface is at $y = h$. The box is assumed to be infinitely extended in the negative y direction. The governing partial differential equation is

$$\begin{aligned} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2(y) \right] \psi(x, y) \\ = -\delta(x - x') \delta(y - y') \end{aligned} \quad (41)$$

where $\psi(x, y)$ is proportional to the z component of the electric field. The solution is given by (for details of the

solution technique, see [14])

$$\psi(x, y) = - \sum_{n=0}^{\infty} X_n(x) X_n(x') Y_n(y, y') \quad (42a)$$

for $-(a/2) < x < (a/2)$, $y < b$, where

$$X_n(x) = \sqrt{\frac{2}{a}} \sin \left[k_{nx} \left(x + \frac{a}{2} \right) \right], \quad -\frac{a}{2} < x < \frac{a}{2} \quad (42b)$$

$$Y_n(y, y') = \begin{cases} \frac{1}{2i\gamma_1} [e^{i\gamma_1|y-y'|} + \tilde{R}_{12} e^{-i\gamma_1(y-h) + i\gamma_1|h-y'|}], & y < h \\ \frac{1}{2i\gamma_1} \tilde{T}_{12} [e^{i\gamma_2(y-h)} - e^{-i\gamma_2(y-2b+h)}], & b > y > h \end{cases} \quad (42c)$$

$$k_{nx} = \frac{n\pi}{a}, \gamma_1 = \sqrt{k_1^2 - k_{nx}^2}, \gamma_2 = \sqrt{k_2^2 - k_{nx}^2}. \quad (42d)$$

In the above, the dependence of γ , $\tilde{R}_{12}$, and $\tilde{T}_{12}$ on n is implied. $\tilde{R}_{12}$ and $\tilde{T}_{12}$ are generalized reflection and transmission coefficients, respectively, at the upper and lower interfaces. They are

$$\tilde{R}_{12} = R_{12} - \frac{T_{12} T_{21} e^{2i\gamma_2(b-h)}}{1 - R_{21} e^{2i\gamma_2(b-h)}} \quad (43a)$$

$$\tilde{T}_{12} = \frac{T_{12}}{1 - R_{21} e^{2i\gamma_2(b-h)}} \quad (43b)$$

where R_{12} and T_{12} are the Fresnel reflection and transmission coefficients. To derive the solution in the complex coordinate system, we make use of Eq. (34), and define

$$s_x(x) = \begin{cases} 1, & 0 < x < x_m \\ 1 + i\alpha \left(\frac{x - x_m}{0.5a - x_m} \right)^2, & x_m < x < \frac{a}{2} \end{cases} \quad (44)$$

where $x_m = 0.5a - \Delta$ and Δ is a small real positive number. In this manner, in accordance with (34), we obtain the mapping

$$\tilde{x} = \begin{cases} x, & 0 < x < x_m \\ x + i\alpha \frac{(x - x_m)^3}{3(0.5a - x_m)^2}, & x_m < x < \frac{a}{2}. \end{cases} \quad (45)$$

In other words, $\tilde{x}$ assumes complex values in this complex coordinate stretching or map. The mapping is linear for the most part, except when $x \approx a/2$. (This map is chosen as an illustration because it resembles the parabolic profile often chosen in PML simulations. Other maps also can be used.) A similar map occurs for $x < 0$. Hence, in the complex coordinate system, the boundary of the box is defined by $\tilde{a}$, which is complex valued. Similar mapping can be defined for the y -axis such that $y \mapsto \tilde{y}$, $b \mapsto \tilde{b}$, where $\tilde{y}$ and $\tilde{b}$ are complex valued.

Figure 1 shows the above solution without complex coordinate stretching. In this case, one clearly sees the reflection

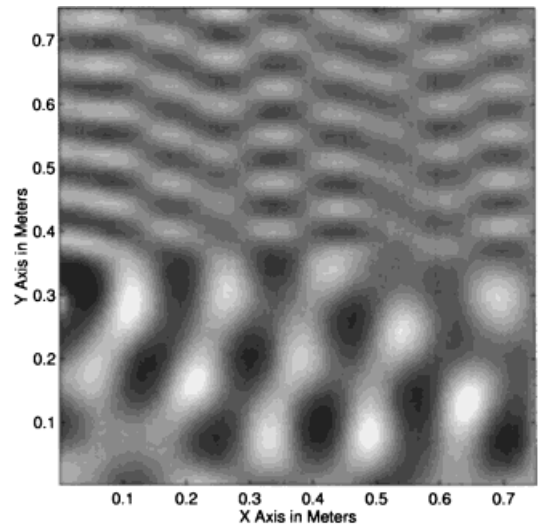


Figure 1 Solution of a line source in a two-dimensional box with $a = b = 1.5$ m. The frequency is 2.2 GHz, $\epsilon_1 = 1$, $\epsilon_2 = 4$, $\alpha = 0$, $\Delta = 0.15$ m, $h = 0.25b$, and $(x', y') = (0, 0.2b)$. One can see many reflections from the walls of the box

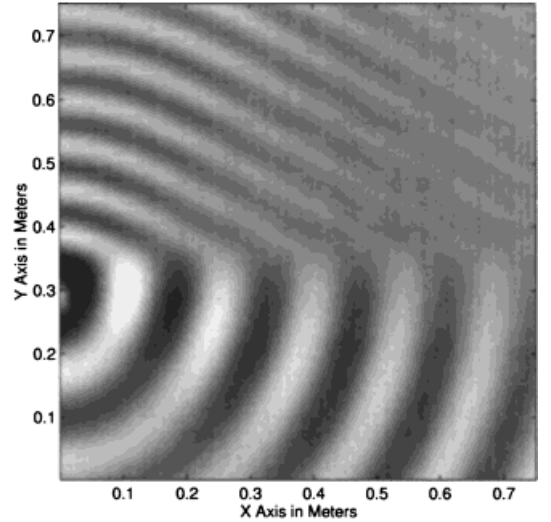


Figure 2 Solution of a line source in a box as before, but in a complex coordinate system with the mapping given by Eq. (45). One can see that the reflection from the wall is greatly diminished. $\alpha = 2$ in the mapping

from the walls at $x = a/2$ and $y = b$. (Only the domain $0 < x < 0.5a$ and $0 < y < 0.5b$ is plotted.) Figure 2 shows the above solution using the complex map provided by Eq. (45). The reflection from the wall is greatly diminished. In this case, as shown by the map, the boundary of the box is located in a complex space. Figure 3 shows a comparison between the closed-form solution $(i/4)H_0^{(1)}(k|\mathbf{p} - \mathbf{p}'|)$ for an infinite space, and the same solution calculated using the series summation in (41) with complex coordinate stretching near the boundary. The plot is for $(x = a/4, 0 < y < b/2)$ and $(0 < x < a/2, y = b/4)$, and (x', y') is the same as the previous figures and $\alpha = 2$. Except for the point at $x = x'$ where the series does not converge well, the agreement between closed form and the series summation with complex coordinate stretching is excellent. Therefore, the complex coordi-

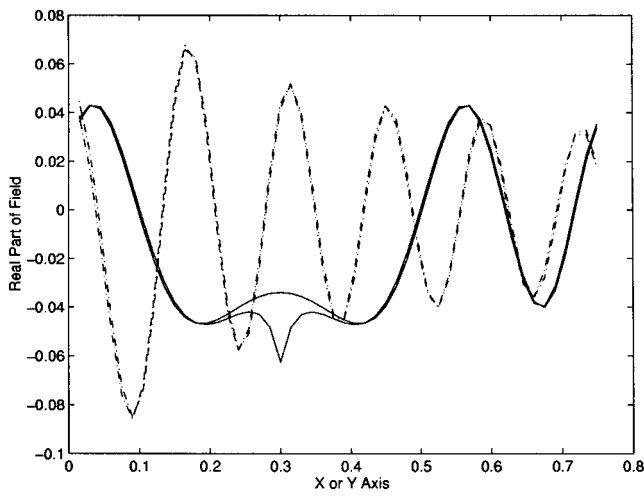


Figure 3 Solution of a line source in a homogeneous medium using a series summation, and the closed-form solution $(i/4)H_0^{(1)}(k|\mathbf{p} - \mathbf{p}'|)$. The fields are plotted at $x = a/4$ and at $y = b/4$. Except for in the vicinity of $y = y'$, where the series summation does not converge well, the agreement is excellent

nate stretching causes the solution of a line source in a box to resemble the solution of a line source in an infinite medium. It also explains why PML remains highly absorptive: 1) when a dielectric interface extends into the PML media, and 2) at the corner region of a simulation box laced with PML.

7. CYLINDRICAL COORDINATES

The wave equation in two-dimensional cylindrical coordinates is

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \psi(\mathbf{p}) = -\delta(\mathbf{p} - \mathbf{p}'). \quad (46)$$

The solution to the above is

$$\psi(\rho, \phi) = \frac{i}{4} H_0^{(1)}(k|\mathbf{p} - \mathbf{p}'|). \quad (47)$$

When a perfectly conducting wall is placed at $\rho = a$, the above solution becomes [14]

$$\psi(\rho, \phi) = \frac{i}{4} H_0^{(1)}(k|\mathbf{p} - \mathbf{p}'|) - \frac{i}{4} \sum_{n=-\infty}^{\infty} J_n(k\rho) \frac{J_n(k\rho') H_n^{(1)}(ka)}{J_n(ka)} e^{in(\phi - \phi')}. \quad (48)$$

We can define a map

$$\tilde{\rho} = \begin{cases} \rho, & 0 < \rho < \rho_m \\ \rho + i\alpha \frac{(\rho - \rho_m)^3}{3(a - \rho_m)^2}, & \rho_m < \rho < a \end{cases} \quad (49)$$

where $\rho_m = a - \Delta$. It can be shown that the reflected field in the above becomes exponentially small under complex coordinate stretching.

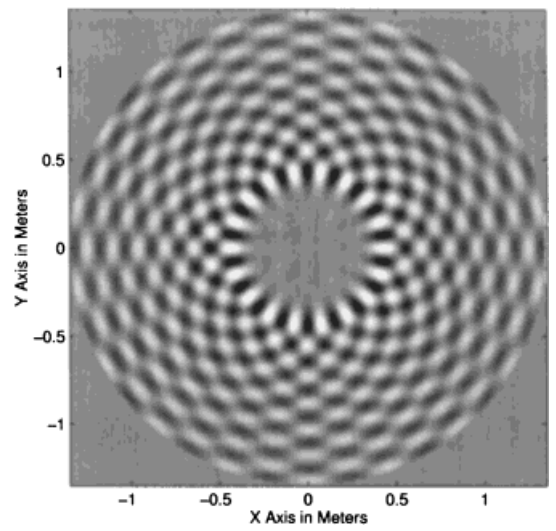


Figure 4 Solution of a line source in a two-dimensional cylinder of radius 1.5 m. The frequency is 2.2 GHz, $\alpha = 0$, and $\Delta = 0.15$ m. One can see many reflections from the wall of the cylinder

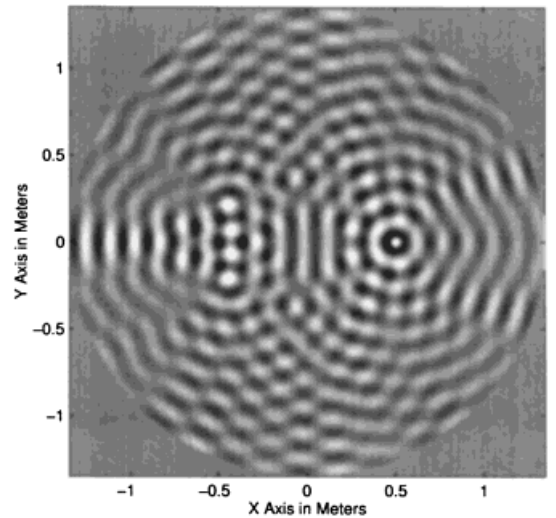


Figure 5 Solution of a line source in a two-dimensional cylinder of radius 1.5 m. The frequency is 2.2 GHz, $\alpha = 0.1$, and $\Delta = 0.15$ m. One can see diminished reflections from the walls of the cylinder

Figures 4–6 show the calculation of the above series summation formula for different values of α . When $\alpha = 0$, there is no complex coordinate stretching at all. When $\alpha > 0$, complex coordinate stretching occurs near the wall according to Eq. (49). The source is placed at $(x', y') = (0.5, 0.0)$ m, the radius of the cylinder is 1.5 m, and the frequency is 2.2 GHz.

8. SPHERICAL COORDINATES

In the spherical coordinates, we have

$$\left(\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + k^2 \right) \psi(\mathbf{r}) = -\delta(\mathbf{r} - \mathbf{r}'). \quad (50)$$

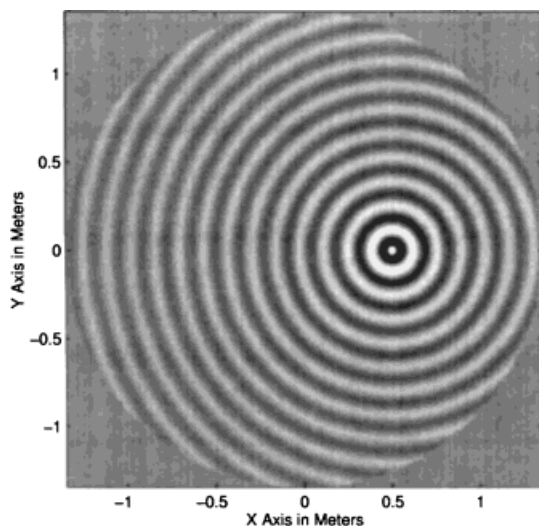


Figure 6 Solution of a line source in a two-dimensional cylinder of radius 1.5 m. The frequency is 2.2 GHz, $\alpha = 2.0$, and $\Delta = 0.15$ m. One can see few reflections from the walls of the cylinder

The solution to the above is

$$\psi(\mathbf{r}) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \quad (51)$$

When a perfectly conducting wall is placed at $r = a$, the solution is [14]

$$\psi(\mathbf{r}) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} - ik \sum_{n=0}^{\infty} \sum_{m=-n}^n j_n(kr) Y_{nm}(\theta, \phi) Y_{nm}^*(\theta', \phi') \frac{j_n(kr') h_n^{(1)}(ka)}{j_n(ka)} \quad (52)$$

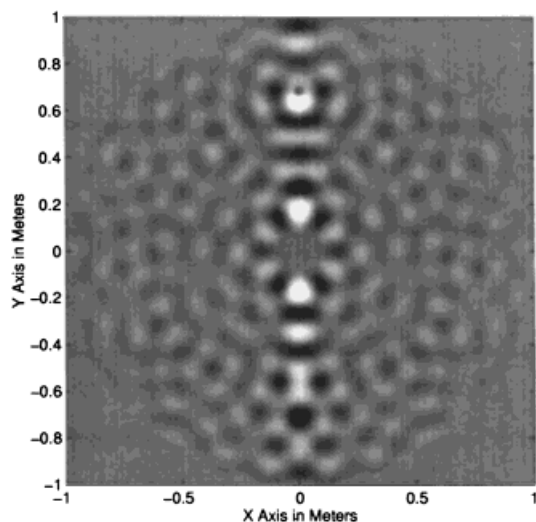


Figure 7 Solution of a point source in a three-dimensional sphere of radius 1.0 m. The frequency is 2.2 GHz, $\alpha = 0$, and $\Delta = 0.15$ m. One can see many reflections from the wall of the sphere

where

$$Y_{nm}(\theta, \phi) = \sqrt{\frac{(n-m)!}{(n+m)!}} \frac{2n+1}{4\pi} P_n^m(\cos \theta) e^{im\phi}. \quad (53)$$

A complex map can be defined such that

$$\tilde{r} = \begin{cases} r, & 0 < r < r_m \\ r + i\alpha \frac{(r-r_m)^3}{3(a-r_m)^2}, & r_m < r < a \end{cases} \quad (54)$$

where $r_m = a - \Delta$. It can be shown that the reflected term becomes exponentially small under the map.

Figures 7–9 show the calculation of the above series summation formula for different values of α . When $\alpha = 0$, there is no complex coordinate stretching at all. When $\alpha > 0$, complex coordinate stretching occurs near the wall according

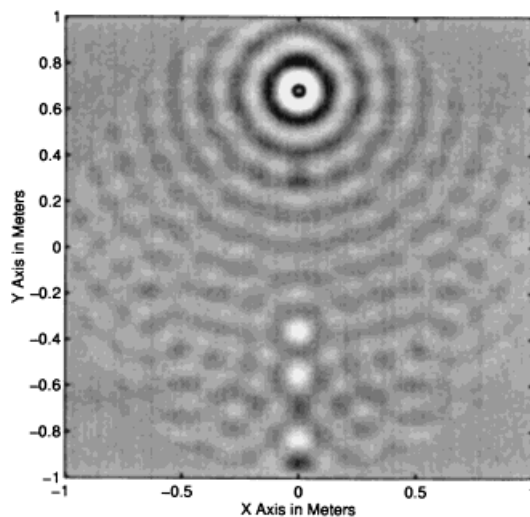


Figure 8 Solution of a line source in a three-dimensional sphere of radius 1.0 m. The frequency is 2.2 GHz, $\alpha = 0.2$, and $\Delta = 0.15$ m. One can see diminished reflections from the walls of the sphere

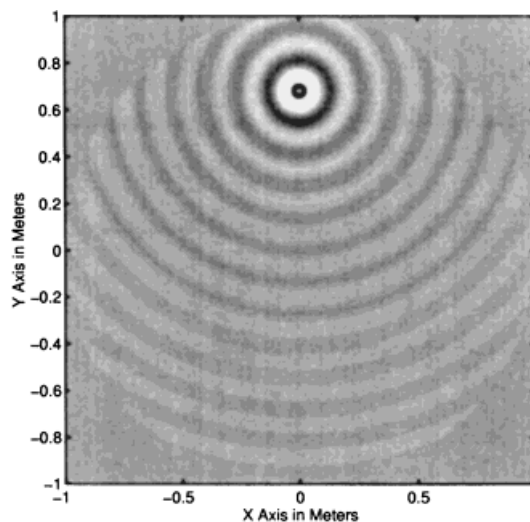


Figure 9 Solution of a line source in a three-dimensional sphere of radius 1.0 m. The frequency is 2.2 GHz, $\alpha = 2.0$, and $\Delta = 0.15$ m. One can see few reflections from the walls of the sphere

to Eq. (54). The source is placed at $(x', y') = (0.0, 0.75)$ m, the radius of the cylinder is 1.0 m, and the frequency is 2.2 GHz.

9. CONCLUSIONS

We have demonstrated that PML is equivalent to complex coordinate stretching, and via a change of variables, PML in Cartesian coordinates is equivalent to solving the wave equation or Maxwell's equations in a complex coordinate system. Therefore, closed-form solutions that exist in the real coordinate system map easily to solutions in the complex coordinate system. In the complex coordinate system, the boundaries can exist in a complex space, providing absorbing boundary conditions. Hence, this transformation provides a new view of PML in the Cartesian coordinates. It also clearly shows that complex coordinate stretching does not induce reflections. Furthermore, it explains why PML works near the corner of a simulation region, and when a dielectric interface, or a metallic surface, extends to the edge of a simulation region.

The idea of complex coordinate stretching, and the fact that it does not induce reflections, can be further generalized to other curvilinear coordinate system. We have demonstrated the solutions and their absorbing boundary conditions in cylindrical as well as spherical coordinates. In this case, one can work backward to derive the actual partial differential equation being solved in the real coordinate system, for instance, by substituting Eq. (49) back into Eq. (46). The end result is a rather complicated looking equation, unlike that in the Cartesian coordinates. However, we can extend the use of finite-difference and finite-element methods directly to complex coordinate systems, and work with partial differential equations with complex coordinates more directly to arrive at absorbing boundary conditions.

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GROUND ERROR CURVES FORMULAS FOR VHF OMNIRANGE RADIO BEACON ANTENNA SYSTEM

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ABSTRACT: A simple method has been introduced for analyzing VHF omnirange radio beacon errors caused by deficiencies in and misalignment of its antenna system. It is based on numerical computation of the Fourier series expansion for a ground error curve, and in so doing, provides a set of formulas for computing the amplitudes of duantal, quadrantal, and octantal errors. © 1997 John Wiley & Sons, Inc. *Microwave Opt Technol Lett* 15: 369–373, 1997.

Key words: VHF omnidirectional range antennas; VOR ground error curves; VOR antennas

1. INTRODUCTION

Field conversions of a conventional VHF omnirange (VOR) radio beacon are used by military and civil aviation for remote maintenance monitoring, which includes, for example, provisions for a fast VOR ground check procedure. The effort is to use a minimum number of ground check points so that the bearing error function can be resolved into its duantal as well as its quadrantal and octantal components.

The ground error curve which is to be taken on a VOR antenna system will be used to assist in analyzing the cause of system misalignment, to recognize deterioration of VOR performance, and to take corrective action. The final adjustment of a VOR installation will depend upon the shape and spread of the ground error curve.

Several basic types of error are classified according to the number of angular positions at which the ground error becomes evident: duantal, quadrantal, and octantal errors. The